Dynamical Systems $\Rightarrow$ possible mode of small amplitude oscillations or waves.

In continuous media simplified problem can be studied by considering plane wave propogation in an infinite homogeneous & time-independent medium.

In an ideal, neutral gas $\Rightarrow$ theory of sound waves

Introduction of more physics like heat, viscosity study decay of sound waves.

Plasma more complicated than ideal gas & more if magnetic field is applied. Properties of plane wave show considerable variety & their investigation reveals many aspects of plasmas.

Homogeneous plasma as continuum point of view

mostly "cold" plasma

For single species in plasma, with number density $n$, velocity $\bar{u}$, pressure $p$, in $\bar{E}$ & $\bar{B}$ fields.

$$\dot{n} + \nabla(n\bar{u}) = 0 \qquad -\ (1)$$

$$nm\frac{D\bar{u}}{Dt} = qn(\bar{E} + \bar{u}\times\bar{B}) - \nabla p \qquad -\ (2)$$

$$\frac{D}{Dt}(p\,n^{-\gamma}) = 0 \qquad -\ (3)$$

$m$ = mass, $q$ = charge, $\gamma$ = ratio of specific heats

& $D/Dt$ = convective rate of change $\partial/\partial t + \bar{u}\cdot\nabla$

No collisions, viscosity or heat conduction.

plasma atleast two species $\Rightarrow$ quasi neutrality
suffix 's' on fluid variables. (eg. $n_s$)
The electric charge density & current

$$\rho = \sum_s n_s q_s \quad , \quad \vec{J} = \sum_s n_s q_s \vec{u}_s \quad - (4)$$

These are source term for Maxwell's eqn

$$\nabla \cdot \bar{B} = 0 \quad - (5)$$

$$\nabla \cdot \bar{E} = \rho / \epsilon_0 \quad - (6)$$

$$\nabla \times \bar{B} = \mu_0 \bar{J} + \epsilon_0 \mu_0 \dot{\bar{E}} \quad - (7)$$

$$\nabla \times \bar{E} = - \dot{\bar{B}} \quad - (8)$$

$$\mu_0 \epsilon_0 = c^{-2}$$

To deal with small pertuobation, assume background as uniform infinite plasma.
The $n, \bar{u}, p, \bar{E}, \bar{B}$ will be denoted as $n_0, \ldots$
also $\bar{u} = \bar{E} = 0$ in unperturbed case. Then $\bar{J} = 0$ & all equ$^n$ 1-8 are satisfied except eq$^n$ (6) which require $\rho = 0$

$$\therefore \sum_s n_s q_s = 0 \quad - (9)$$

condition for neutrality. $\therefore$ for two-speci plasma

$$n_{0i} = n_{0e} = n_0 \quad - (10)$$

Let

$$n = n_0 + n_1$$
$$p = p_0 + p_1 \quad - (11)$$
$$\bar{B} = \bar{B}_0 + \bar{B}_1$$

The other variables vanish. inserting these into (1) - (3) & neglecting second order terms.

$$\dot{n}_1 + n_0 \bar{\nabla} \cdot \bar{u} = 0 \quad — (12)$$

$$n_0 m \dot{\bar{u}} = q n_0 (\bar{E} + \bar{u} \times \bar{B}_0) - \nabla p_1 \quad - (13)$$

$$\frac{p_1}{p_0} = \gamma \frac{n_1}{n_0} \quad — (14)$$

Taking each species to be perfect gas with temp $T$, $p_0 = n_0 k_B T$ eqn (14) becomes

$$p_1 = \gamma k_B T n_1 \quad — (15)$$

To seach for solution of linearized eqn we take perturbation quantities proportional to

$$\exp[i(\omega t - \vec{k} \cdot \vec{x})] \quad — (16)$$

Each vatible is represented ~~by~~ in magnitude & phase $(n_1, \ldots E \ldots)$ by complex amplitude.

physical value of variable is obtained by inserting (16) & taking real part. Thus we replace $\partial/\partial t$ by $i\omega$ & $\nabla$ by $-ik$

The charge conservation eqn

$$\dot{\rho} + \bar{\nabla} \cdot \bar{J} = 0 \quad — (17)$$

Plasma Oscillations

let $\bar{B}_0 = 0$, $T = 0$ consider motion of eletons only & ions at rest $m_i \to \infty$ for electons

$$m_e i\omega \vec{u} = e \vec{E} \quad — (18)$$

$$n_1 = \frac{n_0 e}{m_e \omega^2} \vec{\nabla} \cdot \vec{E} \quad — (19)$$

$$\rho = \frac{n_0 e^2}{m_e \omega^2} \bar{\nabla} \cdot \bar{E} \qquad — (20)$$

Combining with (6)

$$\omega^2 = \frac{n_0 e^2}{m_e \epsilon_0} = \omega_{pe}^2 \qquad — (21)$$

is required. This is plasma oscill$^n$

Elelrons are compressed & rarified like sound waves & restoring fore is $\overline{E}$ rather than $p$

first dispersion relation. in Sound waves $\omega^2 = V_s^2 k^2$ much different. In plasma oscill$^n$ only one frequency is permitted. $\omega_{pe}$ irrespect k

Exercise for ion dynamics $m_i >> m_e$

$$\omega^2 = \omega_p^2 = \omega_{pe}^2 + \omega_{pi}^2 = \omega_{pe}^2 \left(1 + \frac{m_e}{m_i}\right) \qquad — (22)$$

Transverse Waves

Current density & for electron contribution

$-n_0 e \vec{u}$

$$\vec{J} = \frac{n_0 e^2}{m_e i\omega} \vec{E} = \sigma \vec{E} \qquad — (23)$$

$\sigma$ = Ac conductivity in general sum of any term of this type

writing RHS of eq$^n$ (7) as $\mu_0 \dot{D}$

for simple dielectric media.

$$i\omega \bar{D} = \vec{J} + \epsilon_0 i\omega \vec{E} \qquad — (24)$$

or

$$\bar{D} = \epsilon_0 \epsilon \vec{E} \qquad — (25)$$

where $\epsilon$ is the dimensionless dielectic const

$$\epsilon = 1 + \frac{\sigma}{i\omega\epsilon_0} = 1 - \frac{n_0 e^2}{\epsilon_0 m_e \omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2} \quad — (26)$$

We can now write

$$\bar{\nabla} \times \bar{B}_1 = \frac{1}{c^2} \epsilon i\omega \bar{E} \quad , \quad \bar{\nabla} \times \bar{E} = -i\omega \bar{B}_1 \quad — (27)$$

These are identical to those developed in EM waves in simple dielelric (assuming $\epsilon \neq 0$) the Solution has $\bar{E}$ & $\bar{B}$ both perpendicular to $\vec{k}$. The refractive index $N$ is $\epsilon^{1/2}$ so that

$$N = \frac{ck}{\omega} = \left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)^{1/2} \quad — (28)$$

$$k = |\bar{k}|$$

$$\omega^2 = \omega_{pe}^2 + c^2 k^2 \quad — (29)$$

As with EM wave in vacuo, two independent polarization are possible but both satisfy $eq^n$ (28). One choose right-handed & left handed circulary polarized waves. or superpose to make plane polarized.

if ~~$\omega \ll p$~~ $\omega > \omega_{pe}$ , $N < 1$ ; group velocity (parallel to $\bar{k}$) $\partial\omega/\partial k = c^2 k/\omega = cN < c$ in line with special theory of relative the signalling takes place below speed of light.

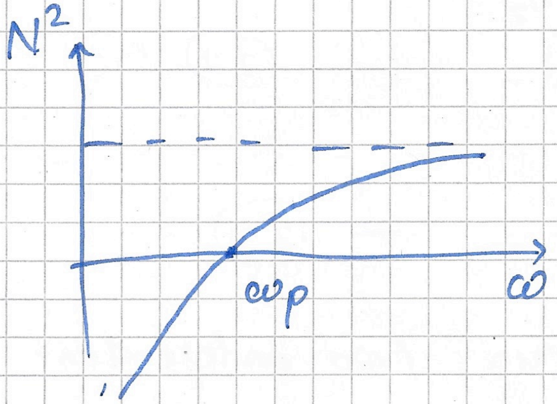
For $\omega < \omega_{pe}$ , $N$ is imaginary then $k = \pm i (\omega_{pe}^2 - \omega^2)^{1/2}/c$. Propagation in the ordinary sense is not possible since exponent $-i\bar{k}\cdot\bar{x}$ is real.

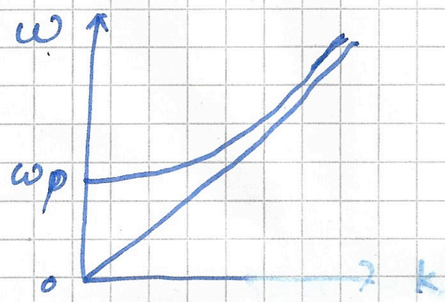
Consider interface $x<0$ = vacuum, in $x>0$ a plasma density $n_0$. EM wave $\omega < \omega_{pe}$ is incident normally from $x<0$, then wave is totally reflected in $x<0$ whereas $x>0$ wave penetrates slightly proportional

$$\exp\left[-(\omega_{pe}^2-\omega^2)^{1/2}x/c\right]$$

$\neq$ such wave is known as "evanescent" & medium is known as "over-dense"

If $\omega \ll \omega_e$ the depth penetration is of order $c/\omega_p$, the plasma wavelength.




Propagation in unmagnetized plasma

$N^2$ as $f^n$ of $\omega$ $\qquad$ $\omega$ as $f^n$ of $k$.

# Effect of pressure

We readmit the term $\nabla p_1$ in eq$^n$ 13 while still omitting $\bar{B}_0$ term. Using eqn 12 & eq$^n$ 13, eliminate $p_1$ fn, we get

$$\vec{u} = \frac{q}{mi\omega}\bar{E} - \frac{\gamma k_B T}{m\omega^2}\nabla(\bar{\nabla}\cdot\bar{u}) \quad — \quad (30)$$

New form of eq$^n$ 18. Using eq$^n$ 30 for both species & allowing $T_i \neq T_e$.

For transverse waves $\bar{E}$, $\vec{u}$ are $\perp$ to $\bar{k}$

so $\bar{\nabla}\cdot\bar{u} = -i\bar{k}\cdot\bar{u} = 0$ $\therefore$ Transverse waves are unaffected by new term in eq$^n$ 30

For Longitudinal waves, all vectors are parallel to $\bar{k}$ & we have one-dim

Using poisson's eq$^n$ & continuity eq$^n$

ie $\bar{E} = -J/\epsilon_0 i\omega$ from eq$^n$ 7

$$E = \frac{ien_0}{\epsilon_0\omega}(u_i - u_e) \quad — \quad (31)$$

& from eq$^n$ 30

$$\left.\begin{aligned} u_i &= \frac{e}{m_i i\omega}E + \frac{\gamma k_B T_i k^2}{m_i\omega^2}u_i \\ u_e &= \frac{-e}{m_e i\omega}E + \frac{\gamma k_B T_e k^2}{m_e\omega^2}u_e \end{aligned}\right\} \quad (32)$$

on the right hand side of eq$^n$ (32) we have restoring forces in form of electrostatic & pressure.
When $e = 0$, two flueds can indulge independent sound waves with $V_{Si}$, $V_{Se}$

$$V_S^2 = \frac{\gamma k_B T}{m} \quad — \quad (33)$$

In general case we eliminate $E$

$$u_i (\omega^2 - \omega_{pi}^2 - V_{Si}^2 k^2) + \omega_{pi}^2 u_e = 0$$
$$u_e (\omega^2 - \omega_{pe}^2 - V_{Se}^2 k^2) + \omega_{pe}^2 u_i = 0 \quad — \quad (34)$$

Elimination of $u_e/u_i$ give dispersion relation which is quadratic for $\omega^2$ in terms of $k^2$ & vice versa.

Let's assume $m_e << m_i$ & $T_e \sim T_i$ not too different so that $V_{Si}$ is comparable to $V_{Se}$. In this case solution to (34) is characterized by $|u_i| << |u_e|$ &

$$\omega^2 \approx \omega_{pe}^2 + V_{Se}^2 k^2 \quad — \quad (35)$$

This is a good approximation provided $k^2$ is not too large.
Meaning eq$^n$ (35): we have electron oscill$^n$ in which electrostatic & pressure combine to make restoring force, so that we have hybrid plasma oscillation & sound wave.

Rewriting eq$^n$ 35

$$\omega^2 \approx \omega_{pe}^2 (1 + \gamma \lambda_D^2 k^2) \quad \text{— } 36$$

$$\lambda_D = \left(\frac{\epsilon_0 k_B T_e}{n_0 e^2}\right)^{1/2} \quad \text{— } 37$$

electron ~~d~~ Debye length. in common case $T_i = T_e$, Debye length is same both species in simple plasma

The second solution to 34, assumink $k$ small

$$\omega^2 \simeq \frac{\omega_{pi}^2 V_{se}^2 + \omega_{pe}^2 V_{si}^2}{\omega_{pe}^2 + \omega_{pi}^2} k^2 = \frac{\gamma k_B (T_e + T_i)}{M_i + m_e} k^2 \quad \text{— } 38$$

is characterized by $u_i \approx u_e$ so $\hat{E} \approx 0$.

This is sound wave, with speed determined by total pressure & total density of two species.

~~Eq~~ Eq$^n$ 36 is rediable provided $\lambda_D k$ is not too large & we take $\gamma = 3 \rightarrow$ collisionless electrons with one degree of freedom.

If $\lambda_D k$ is large, Landau damping is introduced. if $T_e$ is comparable to $T_i$ then Landau damping destroys sound waves eq$^n$ 38

## Cold magnetized plasma

### Dielectric tensor

Propagation in magnetized plama $\rightarrow \bar{B}_0 \neq 0$.

Let cold plama $\rightarrow T = 0$, $p_1 = 0$

from eqn (13) we have

$$m \dot{\bar{u}} = q(\bar{E} + \bar{u} \times \bar{B}_0)$$

we introduce $\exp(i\omega t)$ as before.

Local Dielectric properties depending on $\omega$ but not on spatial structure of wave

$$m i \omega \bar{u} = q(\bar{E} + \bar{u} \times \bar{B}_0) \quad \text{—— (39)}$$

Let $\bar{B}_0 = (0, 0, B_0)$ $\quad B_0 > 0$ &

$$\omega_c = |q B_0 / m|$$

$$i(\omega \pm \omega_c)(u_x \pm i u_y) = \frac{q}{m}(E_x \pm i E_y)$$

$$i \omega u_z = \frac{q}{m} E_z \quad \text{—— (40)}$$

We construct perturbation current $\bar{J} = n_0 q \bar{u}$

$$J_x \pm i J_y = \frac{n_0 q^2}{m i(\omega \pm \omega_c)}(E_x \pm i E_y), \quad J_z = \frac{n_0 q^2}{m i \omega} E_z$$

if $q < 0$ ~~sing~~ signs are interchanged.

Construct vector $\overline{D}$ as defined in eq$^n$ 24

$$(D_x \pm i D_y) = \epsilon_0 \epsilon^{(\pm 1)} (E_x + i E_y),$$
$$D_z = \epsilon_0 \epsilon_0^{(0)} E_z \quad - 41$$

where rea dim less coeff $\epsilon^{\pm 1, 0}$ are defined as

$$\epsilon^{\lambda} = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega + \lambda v_s \omega_{cs})} \quad (\lambda = \pm 1, 0) \quad - 42$$

$v_s = \pm 1$ according to $q \gtrless 0$

for standard singly ionized plasma

since $\omega_{pi}^2 \omega_{ce} = \omega_{pe}^2 \omega_{ci}$

$$\epsilon^{(\lambda)} = 1 - \frac{\omega_p^2}{(\omega + \lambda \omega_{ci})(\omega - \lambda \omega_{ce})} \quad \lambda = \pm 1, 0 \quad - 43$$

for Cartesian components

$$\overline{D} = \epsilon_0 \overline{\epsilon} \, \overline{E} \quad - 44$$

$\overline{\epsilon}$ = dielectric tensor

$$\overline{\epsilon} = \begin{pmatrix} \epsilon^{I} & i\epsilon^{II} & 0 \\ -i\epsilon^{II} & \epsilon^{I} & 0 \\ 0 & 0 & \epsilon^{(0)} \end{pmatrix} \quad - 45$$

$$\epsilon^{I} = \frac{1}{2}(\epsilon^{(1)} + \epsilon^{(-1)}), \quad \epsilon^{II} = \frac{1}{2}(\epsilon^{(1)} - \epsilon^{(-1)}) \quad - 46$$

The medium that follow this form is called "gyrotropic"

To deal with plane waves in gyrotropic medium

curl eq$^n$ of Maxwell's

$$\nabla \times \bar{B}_1 = \frac{\dot{\bar{D}}}{\epsilon_0 c^2} = \frac{i\omega}{c^2} \bar{\epsilon} \cdot \bar{E} , \quad \nabla \times \bar{E} = -i\omega \bar{B}_1$$

eliminating $\bar{B}_1$

$$\nabla \times \nabla \times \bar{E} = \frac{\omega^2}{c^2} \bar{\epsilon} \cdot \bar{E}$$

in tensor notation

$$\left(k^2 \delta_{ij} - \frac{\omega^2}{c^2} \epsilon_{ij}\right) E_j = k_i k_j E_j$$

for non trivial sol$^n$ homogene eq$^n$ E

$$\det\left[k^2 \delta_{ij} - k_i k_j - \frac{\omega^2}{c^2} \epsilon_{ij}\right] = 0$$

This is dispersin relation connecting $\omega$, $\bar{k}$