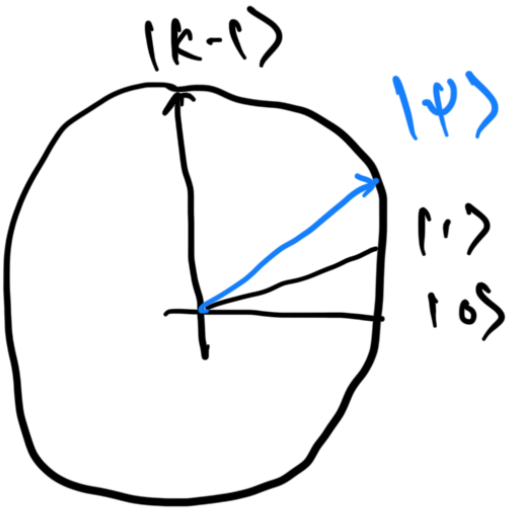
Quantum Mechanics
& Quantum Computing

Quantum gates

Superposition:

Allowable states of k-level system unit vector in k-dim complex vector space (Hilbert space)



$$|\psi\rangle = \alpha_0|0\rangle + \cdots \alpha_{k-1}|k-1\rangle$$

$$= \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{k-1} \end{pmatrix} \in \mathbb{C}^k$$

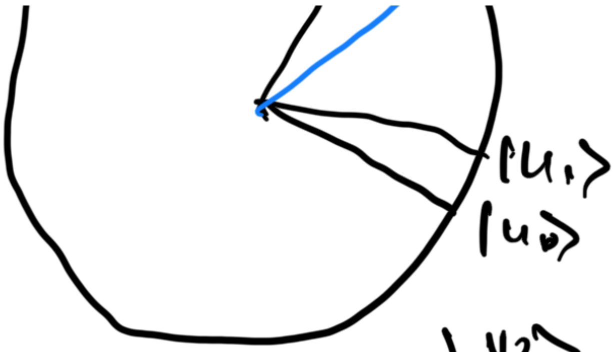
Measurement:

A measurement is specified by choosing an orthonormal basis.

The probability of each outcome is the square of the length of the projection onto corresponding basis vector

The state collapses to the observed basis vector

suppose the chance of result of the vector



$i \rightarrow j$

$j : |(|u_j\rangle, |\psi\rangle)|^2$

$|\psi\rangle = \alpha_0 |0\rangle + \dots + \alpha_{k-1} |k-1\rangle$

$|u_j\rangle = \beta_0 |0\rangle + \dots \quad \beta_{k-1} |k-1\rangle$

inner product

$|(|u_j\rangle, |\psi_j\rangle)| = \bar{\beta}_0 \alpha_0 + \dots \quad \bar{\beta}_{k-1} \alpha_{k-1}$

$$= \begin{pmatrix} \bar{\beta}_0, & \bar{\beta}_1 & \dots & \bar{\beta}_{k-1} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{k-1} \end{pmatrix}$$

Bra-ket notation

$\langle u_j | = (\bar{\beta}_0, \bar{\beta}_1 \dots \bar{\beta}_{k-1})$

inner product $\langle u_j | \psi \rangle$

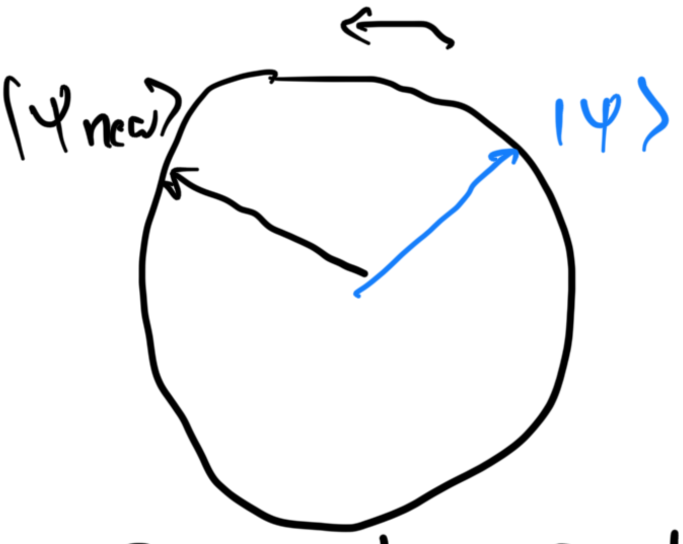
$Pr[j] = |\langle u_j | \psi \rangle|^2$

New state $|\psi'\rangle = |u_j\rangle$

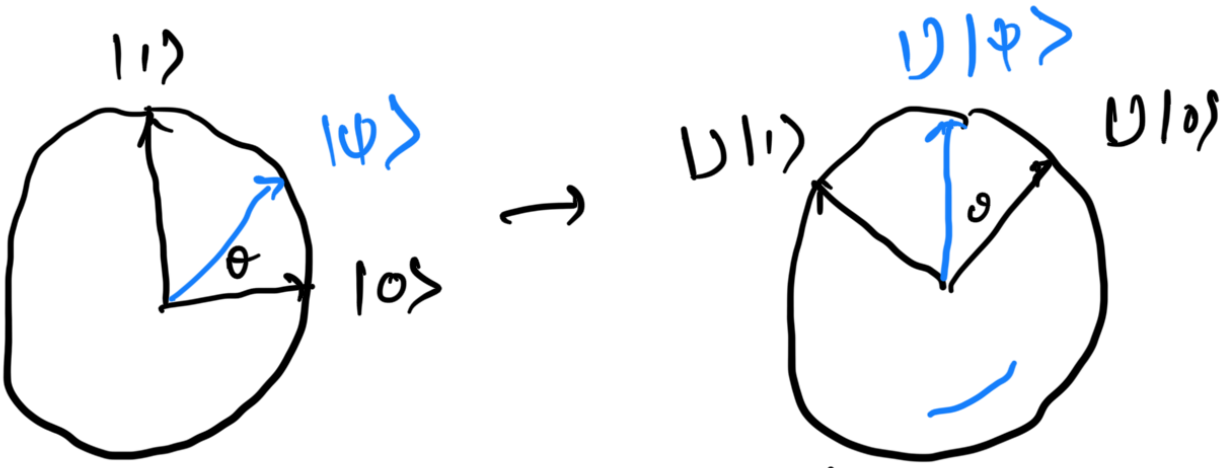
Evolution:

How does the system evolve in time?

By rotation of Hilbert space

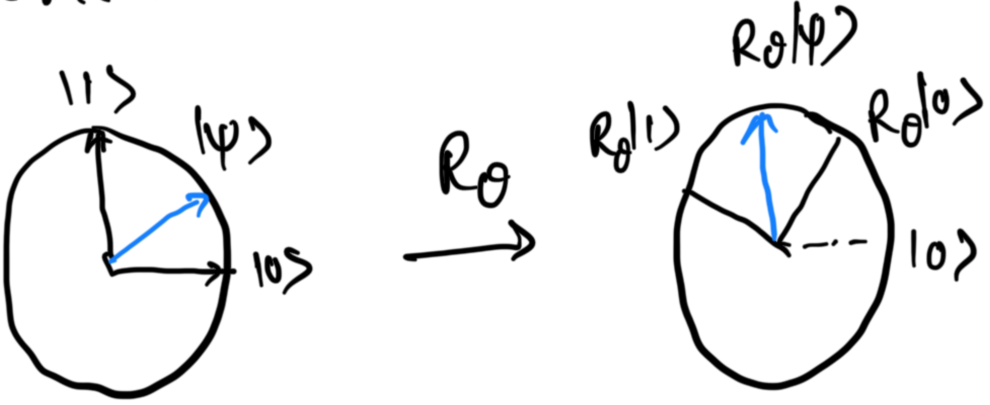


Example: Evolution of a qubit in 2D



angles are preserved

Rotation of the space is given by linear transformation, Matrix:



$$|0\rangle \xrightarrow{R_\theta} \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$$

$$|1\rangle \xrightarrow{R_\theta} -\sin\theta\,|0\rangle + \cos\theta\,|1\rangle$$

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$R_{-\theta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = R^T$$

$$R_\theta R_{-\theta} = R_{-\theta} R_\theta = I$$

$$\boxed{R_\theta R_\theta^T = R_\theta^T R_\theta = I}$$

unitary transformation

in $\mathbb{C}^K$ vector space $K \times K$ matrix

$$U = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

complex conjugate

& transpose

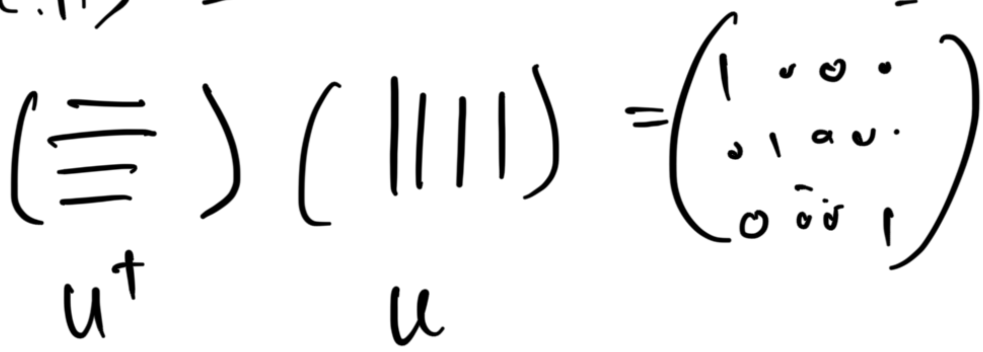
$U^\dagger U = UU^\dagger = I \iff U$ is unitary

$U = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad U^\dagger = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$

$\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$U|0\rangle = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} = a|0\rangle + b|1\rangle$

$U|1\rangle = \qquad = c|0\rangle + d|1\rangle$



U preserves inner product:

$\langle \phi | \psi \rangle = \langle \phi | U^\dagger U | \psi \rangle$

Quantum Gates:

$|\psi\rangle \longrightarrow \boxed{U} \longrightarrow U|\psi\rangle$

wire $\leftrightarrow$ qubit $\rightarrow$ unitary transform

"Bit flip"

$\alpha_0|0\rangle + \alpha_1|1\rangle \longrightarrow \boxed{X} \longrightarrow \alpha_1|0\rangle + \alpha_0|1\rangle$

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$

Similarly $X|1\rangle = |0\rangle$

verify X is unitary

$$X^{\dagger}X = XX^{\dagger} = I = X^2$$

"Phase flip"

$$\alpha_0|0\rangle + \alpha_1|1\rangle \longrightarrow \boxed{Z} \longrightarrow \alpha_0|0\rangle - \alpha_1|1\rangle$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad Z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$Z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -|1\rangle$$

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$Z|+\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = -|-\rangle$$

$$Z|-\rangle = |+\rangle$$

verify Z is unitary

"Hadamard transform"

$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \qquad \begin{aligned} |0\rangle &\to H|0\rangle = |+\rangle \\ |1\rangle &\to H|1\rangle = |-\rangle \end{aligned}$$

$$H^{\dagger}H = \,?$$ unitary

All in three gate if one applies gate twice initial state is reached

$$|0\rangle \xrightarrow{X} |1\rangle$$

$|0\rangle \qquad\qquad\qquad |1\rangle$

$H \Big\downarrow \qquad\qquad\qquad \Big\uparrow H$

$|+\rangle \xrightarrow[\quad Z \quad]{} |-\rangle$

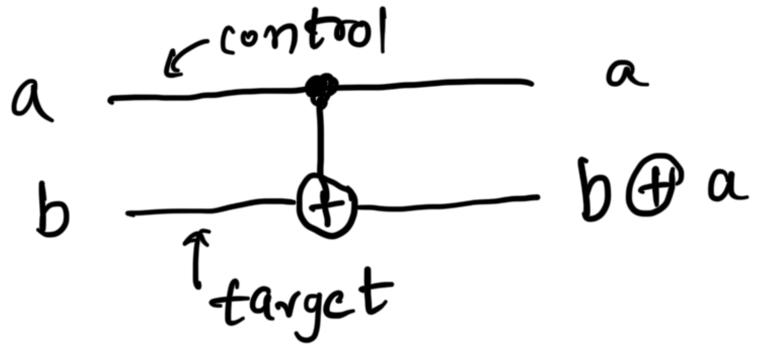
$$X = HZH$$

Two qubit gates

$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle \ldots \Rightarrow \boxed{U} \Rightarrow \alpha'_{00}|00\rangle + \alpha'_{01}|01\rangle \ldots$

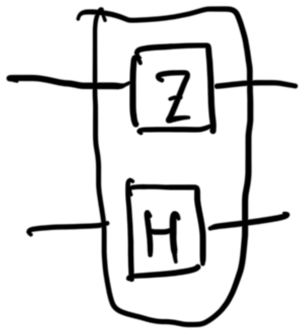
U is 4×4 matrix

"CNOT gate"

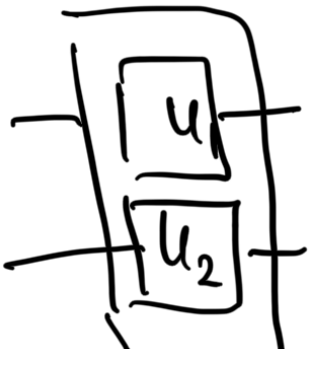


$|00\rangle \to |00\rangle$
$|01\rangle \to |01\rangle$
$|10\rangle \to |11\rangle$
$|11\rangle \to |10\rangle$

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \\ & & & \\ 0 & & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ & & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$



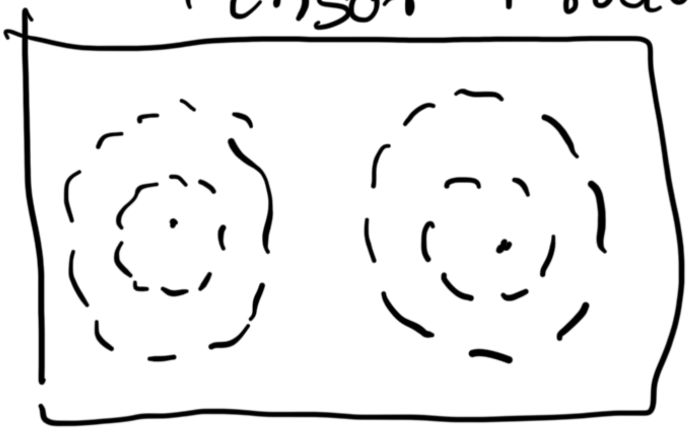
$$U_1 = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\begin{array}{ccc} & 00 & 11 \\ 00 & & \\ U = & \begin{pmatrix} a\begin{pmatrix} e & g \\ f & h \end{pmatrix} & c\begin{pmatrix} e & g \\ f & h \end{pmatrix} \\ \cdots & \cdots \end{pmatrix} & \end{array}$$

$U_2 = \begin{pmatrix} e & g \\ f & h \end{pmatrix}$

$\begin{bmatrix} b\begin{pmatrix} e & g \\ f & h \end{pmatrix} & d\begin{pmatrix} e & g \\ f & h \end{pmatrix} \end{bmatrix}$

Tensor Product



$$\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$$

$$H_1 \otimes H_2 = H$$

$|u\rangle \quad |v\rangle \quad |u\rangle \otimes |v\rangle$

$|u_1\rangle + |u_2\rangle \quad |v\rangle \quad |u_1\rangle \otimes |v\rangle + |u_2\rangle \otimes |v\rangle$

$|0\rangle \otimes |0\rangle = |0\rangle|0\rangle = |00\rangle$

bilinearit

entangled state.

$|u_1\rangle \otimes |v_1\rangle \quad |u_2\rangle \otimes |v_2\rangle$

inner product

$= \langle u_1|u_2\rangle \langle v_1|v_2\rangle$