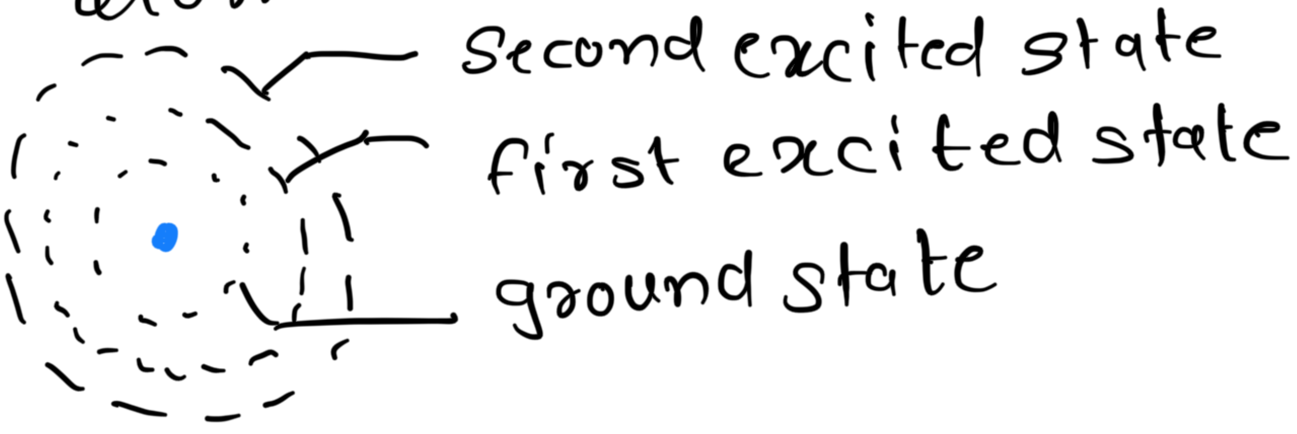
Quantum Mechanics & Quantum Computing

Axioms of QM, two Qubits & Entanglement

Energy of an electron in an atom



Electron can not take any value, must be in discrete energy level. $0 \to k-1$ excited state. If electron is classical system then it could store k-bits / pieces of information

$|0\rangle, |1\rangle \dots |k-1\rangle$.

Superposition: general state is linear superposition of $|0\rangle$ to $|k-1\rangle$

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle \dots + \alpha_{k-1} |k-1\rangle$$

$$\alpha_j \in \mathbb{C}$$

$$\sum^{k-1} |\alpha_j|^2 = 1$$

$\sum_{j=0} \alpha_j$

Measurement Axiom: when we measure a system or "look" at it, the probability that outcome is j:

$P[j] = |\alpha_j|^2$ since the state is normalized, we get probability 1 we get some outcome betn j betn 0 & $k-1$. Measurement disturbs the system & new state

$|\psi'\rangle = |j\rangle$.

example: $k = 3$

$$|\psi\rangle = \left(\frac{1}{2} + \frac{i}{2}\right)|0\rangle + \frac{1}{2}|1\rangle + \frac{i}{2}|2\rangle$$

Measure

$P[0] = \frac{1}{2}$, $|\psi'\rangle = |0\rangle$

$P[1] = 1/4$, $|\psi'\rangle = |1\rangle$

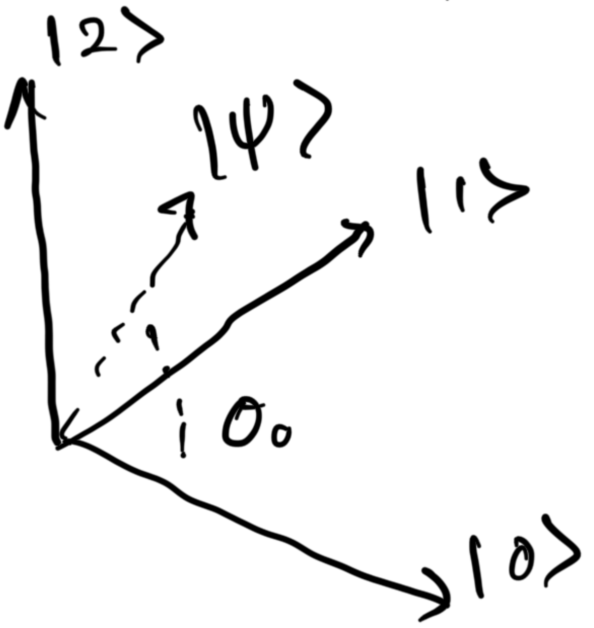
$P[2] = 1/4$, $|\psi'\rangle = |2\rangle$

Geometric interpretation:

Superposition principle:

$|\psi\rangle \in \mathbb{C}^k$ state of a quantum system is a unit vector in k-dimension complex vector space

state is a unit vector in a Hilbert space $\mathbb{C}^k$.



$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \alpha_2|2\rangle$$

$$= \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} \in \mathbb{C}^3$$

when we measure, in standard basis

$P[0] = \cos^2\theta_0$ state vector is projected

How do we define θ

$$|\psi\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} \quad |\phi\rangle = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}$$

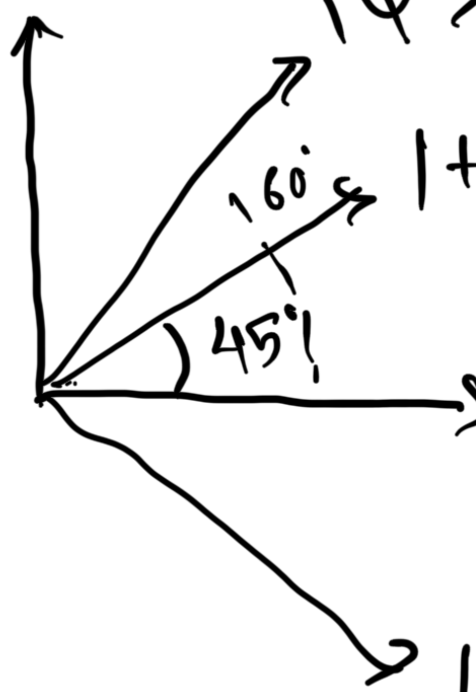
inner product $= \alpha_0^* \beta_0 + \alpha_1^* \beta_1 + \alpha_2^* \beta_2$

$$= (\alpha_0^* \ \alpha_1^* \ \alpha_2^*) \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}$$

$$\cos\theta = |\alpha_0^* \beta_0 + \alpha_1^* \beta_1 + \alpha_2^* \beta_2|$$

magnitude of complex number

Example:



$$|\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$|0\rangle$

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$P[+] = \left(\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} \end{pmatrix} \right)^2$$

$$= \left(\frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}} \right)^2$$

$$= \left(\frac{1+\sqrt{3}}{2\sqrt{2}} \right)^2$$

$$= \frac{2+\sqrt{3}}{4}$$

$$P[-] = \frac{2-\sqrt{3}}{4}$$

measurement in sign basis

$$|\psi\rangle = \alpha\,|+\rangle + \beta\,|-\rangle$$

$$|0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

$$|1\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle$$

$$|\psi\rangle = \frac{1}{2}\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right) + \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle\right)$$

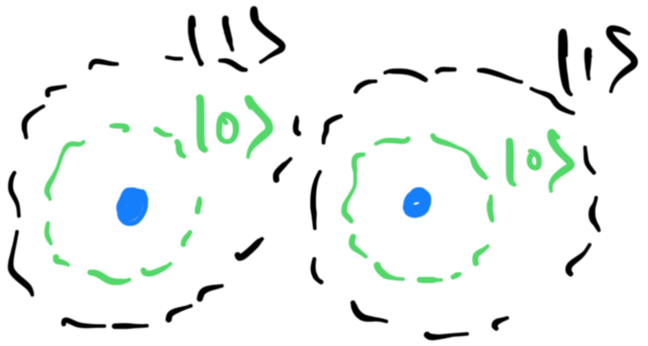
$$= \left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)|+\rangle + \frac{1-\sqrt{3}}{2\sqrt{2}}|-\rangle$$

$$P\,|+\rangle = \frac{1+3+2\sqrt{3}}{8} = \frac{2+\sqrt{3}}{4}$$

System of two Qubits :

H-atom : ground or excited state , two such electrons

Classically four state 00 , 01 , 10 , 11



$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle +$$

$$\alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

$$\alpha_x \in \mathbb{C}$$

$$\sum |\alpha_x|^2 = 1 \quad \text{normalized}$$

Measure:

$$P[00] = |\alpha_{00}|^2$$

New state $= |\psi'\rangle = |00\rangle$

$$P[01] = |\alpha_{01}|^2$$

$$\vdots$$

Example:

$$\psi = \left(\frac{1}{2} + \frac{i}{2}\right)|00\rangle + \frac{1}{2}|01\rangle - i/2\,|11\rangle$$

$$P[00] = 1/2$$

$$P[01] = 1/4$$

$$P[11] = 1/4$$

Partial measurement: what is the result of measuring just first qubit

$$P[0] = |\alpha_{00}|^2 + |\alpha_{01}|^2$$

New state:

$$|\psi'\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle$$

$$|\psi'\rangle = \frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}}$$

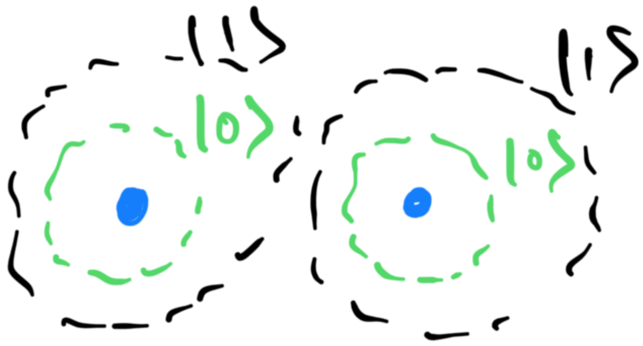
normalization

$$P[0] = \underline{\underline{\tfrac{1}{2}}} + \underline{\underline{\tfrac{1}{4}}} = 3/4$$

$$|\psi'\rangle = \frac{\left(\frac{1}{2} + \frac{i}{2}\right)|00\rangle + \frac{1}{2}|01\rangle}{\sqrt{3/4}}$$

Entanglement:

two qubit system



composit system

$\alpha_0 |0\rangle + \alpha_1 |1\rangle$ $\quad$ $\beta_0 |0\rangle + \beta_1 |1\rangle$

we are given state of the each qubit individually

$$= \alpha_0\beta_0 |00\rangle + \alpha_0\beta_1 |01\rangle + \alpha_1\beta_0 |10\rangle + \alpha_1\beta_1 |11\rangle$$

suppose $\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$

$$\left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right)$$

composite system state:

$$\frac{1}{2\sqrt{2}}|00\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|01\rangle + \frac{1}{2\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|11\rangle$$

But we can not determine individual state from any given arbitrary composite state.

Consider $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

$= (\alpha_0|0\rangle + \alpha_1|1\rangle)(\beta_0|0\rangle + \beta_1|1\rangle$

$= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle$
$+ \alpha_1\beta_1|11\rangle$

$\alpha_0\beta_0 = 1/\sqrt{2} = \alpha_1\beta_1$

non zero non zero

But $\alpha_0\beta_1$ & $\alpha_1\beta_0$ must be zero

i.e. α_0 or β_1 is zero
or α_1 or β_0 is zero.

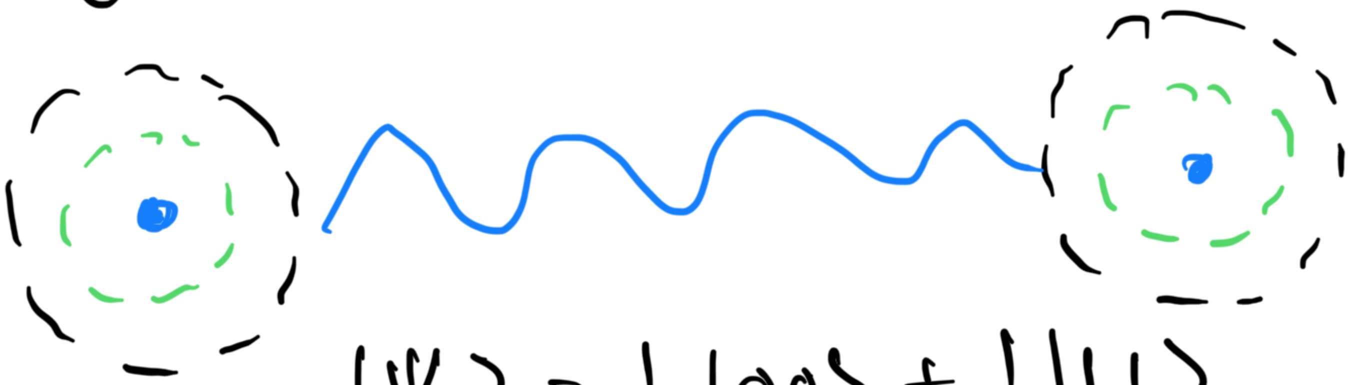
Contradiction

We can not factor state
as state 1 times state 2.

Quantum systems have property
that bring two system
together & let interact,
they get into state that
neither of the two quantum
systems described by itself.
They get "entangled" such

that only way to describe it is all of it.

This entangled state persist even after we separate.



$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Measuring Bell state two state now separated

$P[0] = 1/2 \quad |\psi'\rangle = |00\rangle$

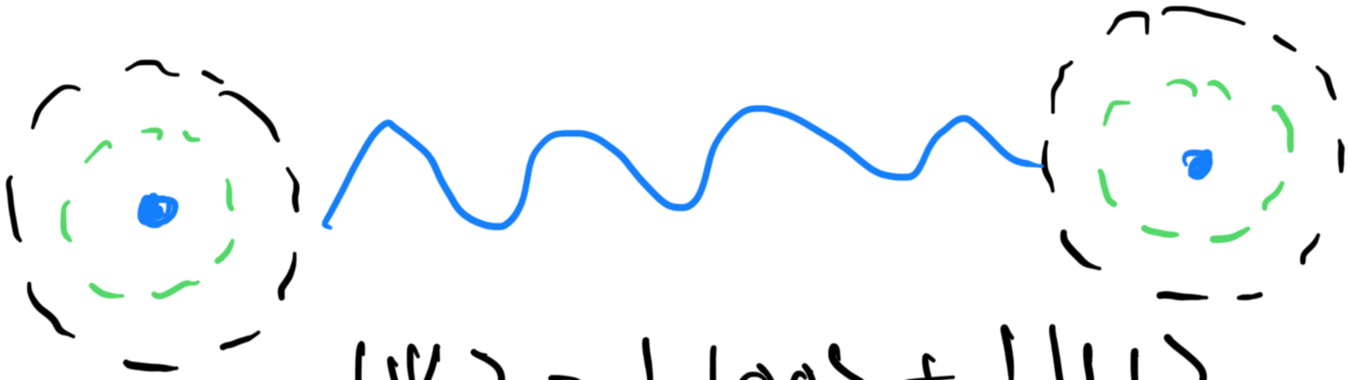
$P[1] = 1/2 \quad |\psi'\rangle = |11\rangle$

second qubit by symmetry

$P[0] = 1/2$

what if we measure first qubit zero ($|0\rangle$) with new state $|\psi'\rangle = |00\rangle$ when we measure second qubit it certainly zero state no matter how far the qubits are.

EPR Paradox:



$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

$$= \frac{1}{\sqrt{2}}|++\rangle + \frac{1}{\sqrt{2}}|--\rangle$$

election both in ground state or both in excited state

state of each electron in +, - basis

$$\frac{1}{\sqrt{2}}|++\rangle + \frac{1}{\sqrt{2}}|--\rangle$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$+ \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left[\,|00\rangle + |11\rangle\,\right]$$

Bell state.

sign: $P[+] = \frac{1}{2}$

$P[-] = \frac{1}{}$

again if first qubit in state + then second is certainly in state + & vice versa.

Einstein-Podolsky-Rosen (1935) formulated the paradox showing dissatifaction with QM!

One can choose to measure the first qubit in Bit-basis or sign basis. if we chose Bit basis, we know Bit value of first then we know Bit value of second due to Bell state

if we choose sign. we then know the sign of the second. Second qubit is not affected by which value one decides to measure. Thus both values of the second qubit is already determined. This is contradictory to uncertainty principle. Even if we measure the bit value of the first qubit it disturbs the sign value.

But does not change the sign value of the second qubit.