Lecture 6: The Ising Model and Collective Ordering

6.1 Spins and interactions

The Ising model is a minimal statistical-mechanical description of interacting binary variables. Each lattice site carries a spin $s_i = +1$ or -1. Nearest-neighbor spins interact, and an external magnetic field may favor one orientation.

$$\boxed{H = -J \sum_{\langle i,j \rangle} s_i s_j - h \sum_i s_i.} \tag{1}$$

For $J > 0$, aligned neighboring spins lower the interaction energy; this is the ferromagnetic case. The external field h favors positive or negative magnetization.

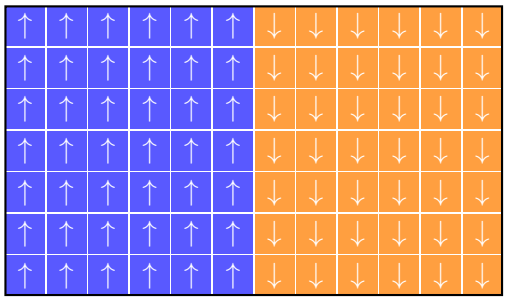


Figure 6. A schematic boundary between two aligned spin domains.

6.2 Temperature competes with ordering

At low temperature, configurations with low energy are strongly favored. At high temperature, thermal fluctuations can disrupt alignment. The Boltzmann weight of a configuration is proportional to

$$P(\{s_i\}) \propto e^{-H/(k_B T)}. \tag{2}$$

We do not calculate the partition function in this course. The expression is included only to show the competition between interaction energy and thermal fluctuations.

The magnetization per spin is

$$m = \frac{1}{N} \sum_{i=1}^{N} s_i. \tag{3}$$

For an ordered ferromagnet, $|m|$ can be appreciable. In a disordered state with balanced up and down spins, m is near zero.

6.3 Phase transitions and domains

For the infinite, zero-field, nearest-neighbor ferromagnetic Ising model on a two-dimensional square lattice, a finite critical temperature separates ordered and disordered phases. Finite systems show rounded transitions rather than ideal mathematical singularities. In one dimension with only short-range interactions, the corresponding model has no finite-temperature transition.

Domains are regions where many spins share an orientation. Domain boundaries cost interaction energy, while entropy favors a greater variety of configurations. Their competition is central to collective ordering.

6.4 Why include the Ising model?

The Ising model illustrates how simple microscopic interactions produce a macroscopic order parameter. It also introduces the language of energy landscapes used in associative-memory networks. The Ising model is primarily an *equilibrium statistical-mechanics model*, unlike the externally driven BZ reaction or laser.

Key idea. Many locally interacting binary variables can collectively develop order, even though no individual spin controls the entire system.

Further reading

K. Huang, *Statistical Mechanics*; H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena*.