Magnetohydrodynamics (MHD)

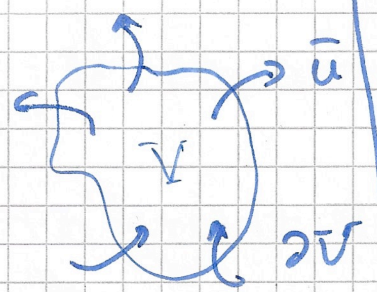
A relatively simple theory when compared to Kinetic & fluid theory, rather perhaps crude
It can describe rich & varied phenomena
Used in

- Finding magnetic field confugurations capable of confining plasma in equilibrium
- linear stability properties of equilibrium
- non-linear development of instabilities & consequences.

MHD can be derived from Vlasov-Maxwell-Landau equ^ns in the limit of large collisionality & additional assumptions.
Fluid approach. for consistency

We have already seen it consits of eq^n ~~of~~ for: conservation of mass, momentum & energy along with Maxwell's eq^n,
Consider fluid characterized. by

ρ - mass density
$\bar{u}$ - flow velocity
p - pressure
σ - charge density
$\bar{j}$ - current density
$\bar{E}$ - electric field
$\bar{B}$ - magnetic field.



Continuity eq^n

$$\frac{\partial \rho}{\partial t} + \nabla(\rho u) = 0 \quad \text{— (1)}$$

flowins in-out of Volum V enclosed by ∂V

$$\underbrace{\frac{d}{dt}\int_V d^3\bar{r}\, \rho \bar{u}}_{\text{momentum inside Volume of fluid}} = -\int_{\partial V} \underbrace{\rho \bar{u}\bar{u}\cdot d\bar{s}}_{\substack{\text{Reynold stress} \\ \text{momentum flux through boundary} \\ \text{carry own momentum}}} - \underbrace{\int p\, d\bar{s}}_{\substack{\text{pressure} \\ \text{on boundary}}}$$

$$- \underbrace{\int_{\partial V} \bar{\Pi}\cdot d\bar{s}}_{\text{Viscous stress}} + \underbrace{\int_V d^3\bar{r}\, \bar{F}}_{\substack{\text{all other forces} \\ E, M}} \quad \text{— (2)}$$

$$= \int_V d^3\bar{r} \left[-\nabla\cdot(\rho\bar{u}\bar{u}) - \nabla p - \nabla\cdot \Pi + F \right]$$

in differential form.

$$\underbrace{\rho\left(\frac{\partial \bar{u}}{\partial t} + \bar{u}\cdot\nabla \bar{u}\right)}_{= \frac{d\bar{u}}{dt}} = -\nabla p - \nabla\cdot \Pi + \bar{F} \quad \text{— (3)}$$

convective derivative.

This is momentum equ^n.

The conducting fluid → we have distributed charges (σ) & currents ($\bar{j}$) so that electric field ($\bar{E}$) & magnetic field ($\bar{B}$) will exert body forces on fluid for single particle

$$\bar{f}_L = q\left(\bar{E} + \frac{\bar{v}\times\bar{B}}{c}\right) \quad — (4)$$

So summing over all particles or precisely average over distribution & sum over all species

$$\bar{F} = \sigma\bar{E} + \frac{\bar{j}\times\bar{B}}{c} \quad — (5)$$

This is to be substituted in eqn (3)

Maxwell eqn

$$\nabla\cdot\bar{E} = 4\pi\sigma \quad \text{(Gauss)} \quad — (6)$$

$$\nabla\cdot\bar{B} = 0 \quad — (7)$$

$$\frac{\partial B}{\partial t} = -c\nabla\times\bar{E} \quad \text{(faraday)} \quad — (8)$$

$$\nabla\times\bar{B} = \frac{4\pi}{c}\bar{j} + \frac{1}{c}\frac{\partial\bar{E}}{\partial t} \quad — (9)$$

Ampere-Maxwell

Append with Ohm's law in simplest form:

Electric field in the fram of fluid element moving with velocity $\bar{u}$ is

$$\bar{E}' = \bar{E} + \frac{\bar{u}\times\bar{B}}{c} = \eta\bar{j} \quad — (10)$$

↓ lab frame ↓ ohmic resistivity.

let's assume non-relativistic case & let's stipulate that all fields ~~[illegible]~~ evolve on time scale $\sim \tau$, & have spatial scale $\sim l$
~~that the~~ fluid velocity

$$u \sim \frac{l}{\tau} << c \quad — (11)$$

~~Then~~

$$E \sim \frac{u}{c} B << B \quad — (12)$$

$\therefore$ electric fields are small compared to magnetic fields.

So in ampere-maxwell's law

$$\frac{\left|\frac{1}{c}\frac{\partial \bar{E}}{\partial t}\right|}{|\nabla \times B|} \sim \frac{\frac{1}{c}\frac{1}{\tau}\frac{u}{c}B}{\frac{1}{l}B} \sim \frac{u^2}{c^2} << 1 \quad — (13)$$

$\therefore$ displacement current is negligible.

$$j = \frac{c}{4\pi} \nabla \times \bar{B} \quad — (14)$$

$\therefore$ $\bar{J}$ & $\bar{B}$ have one to one correspondece. & not independent.

Comparing eletric & magnetic fields part of lorentz eqn & usin Gauss's Law to estimate $\sigma \sim E/l$

$$\left|\frac{|\sigma \bar{E}|}{|\frac{1}{c}\bar{J} \times \bar{B}|}\right| \sim \frac{\frac{1}{l}E^2}{\frac{1}{c}\frac{c}{l}B^2} \sim \frac{E^2}{B^2} \sim \frac{u^2}{u^2} << 1$$

(15)

Thus MHD body force

$$F = \frac{\bar{j} \times \bar{B}}{c} = \frac{(\bar{\nabla} \times \bar{B}) \times B}{4\pi} \quad -(16)$$

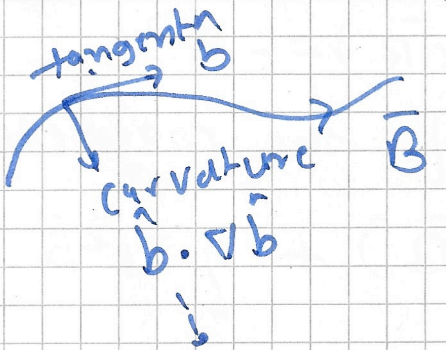
with simple vector algebra.

$$\bar{F} = \underbrace{\frac{\bar{B} \cdot \nabla B}{4\pi}}_{\text{Magnetic Tension.}} - \underbrace{\nabla \frac{B^2}{8\pi}}_{\text{Magnetic pressure}} = -\nabla \cdot \underbrace{\left(\frac{B^2}{8\pi} I - \frac{\bar{B}\bar{B}}{4\pi} \right)}_{\text{Maxwell stress}} \quad -(17)$$

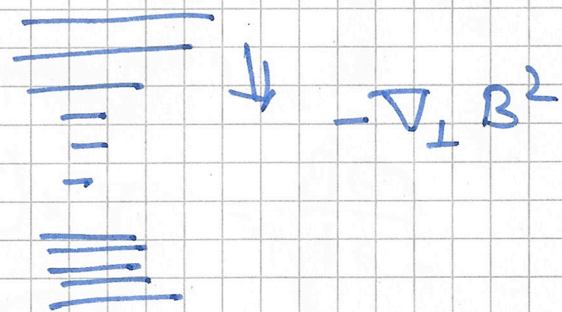
Maxwell stress $\approx$ suspension in fluid of elongate molecule

or ball on spring.

if we rewrite magnetic tension & pressure force as: let $\bar{b} = \bar{B}/B$ unit vector in direction of $\bar{B}$ then



Curvature force



magnetic pressure

$$\bar{B} \cdot \nabla \bar{B} = B\, b \cdot \nabla (B \bar{b}) = B^2 b \cdot \nabla b + \bar{b} \bar{b} \cdot \nabla \frac{B^2}{2} \quad -(18)$$

$$\bar{F} = \underbrace{\frac{B^2}{4\pi} \bar{b} \cdot \nabla \bar{b}}_{\text{curvature force.}} - \underbrace{(I - \bar{b}\bar{b}) \cdot \nabla \frac{B^2}{8\pi}}_{= \nabla_\perp B \text{ pressure force}} \quad -(19)$$

- curvature force: $\bar{b}\cdot\bar{\nabla} b \rightarrow$ curvature of field lines – implication being that field lines, if bent, will want to straighten up

- magnetic pressure: whose presence implies that field line will try to resist compression or rarification.

Note both forces act perpendicular to $\bar{B}$ As they should because magnetic field itself does not exert force on charged particles.

This is effect of field on fluid.

Using ohm's law we write

$$\frac{\partial B}{\partial t} = \nabla \times \left(\bar{u} \times \bar{B} - \frac{c^2 \eta}{4\pi} \bar{\nabla} \times \bar{B} \right) \quad - (20)$$

$\nabla \cdot \bar{B} = 0 \Rightarrow \quad \nabla \times (\bar{\nabla} \times \bar{B}) = -\nabla^2 B$

& $\quad c^2\eta/4\pi \rightarrow \eta$ the magnetic diffusivity

$$\frac{\partial \bar{B}}{\partial t} = \underbrace{\nabla \times (\bar{u} \times \bar{B})}_{\text{advection}} + \underbrace{\eta \nabla^2 \bar{B}}_{\text{diffusion}} \quad - (21)$$

magnetic induction eqn.

if $\bar{\nabla}\cdot\bar{B}$ is satisfied initially solution to eqn (21) will remain divergence free.

The ratio of advective & diffusive term

$$\frac{|\nabla \times (\bar{u} \times \bar{B})|}{|\eta \bar{\nabla}^2 B|} \sim \frac{\frac{u}{l} B}{\frac{\eta}{l^2} B} = \frac{ul}{\eta} \equiv R_m \quad \text{— (22)}$$

This is dimensionless number Magnetic Reynold's number.

liquid metals metallurgy : $10^{-3} \cdots 10^{-1}$

planets interior : $100 - \cdots 300$

interstellar medium : 10^{18}

solar convection zone : $10^{6} - 10^{9}$

core of galaxy : 10^{29}

when flow velocities are large / distances are large / resistivity is low $R_m \gg 1$

it is idea MHD ie $\lim \eta \to 0$

$\therefore \eta \to 0$

$$\frac{\partial \bar{B}}{\partial t} = \nabla \times (\bar{u} \times \bar{B}) \quad \text{— (23)}$$

implies that fluid element that lie on a field lines initially will remain on this field line ie "magnetic field moves with flow" $\longrightarrow$ Lundquist theorem.

using Convective derivative & unpacking bracket terms

$$\frac{d\bar{B}}{dt} = \left(\frac{\partial}{\partial t} + \bar{u}\cdot\nabla\right)\bar{B} = \bar{B}\cdot\nabla\bar{u} - B\,\nabla\cdot\bar{u} \quad \text{— (24)}$$

Continuity eqn

$$\frac{d\rho}{dt} = \left(\frac{\partial}{\partial t} + \bar{u}\cdot\nabla\right)\rho = -\rho\,\nabla\cdot\bar{u}$$

$$\Rightarrow \quad \nabla\cdot\bar{u} = -\frac{1}{\rho}\frac{d\rho}{dt} \quad \text{— (25)}$$

using expression of $\nabla\cdot\bar{u}$

$$\frac{dB}{dt} = \bar{B}\cdot\nabla u + \frac{\bar{B}}{\rho}\frac{d\rho}{dt} \quad \text{— (26)}$$

Multiply in and rearranging

$$\frac{d}{dt}\left(\frac{\bar{B}}{\rho}\right) = \frac{\bar{B}}{\rho}\cdot\nabla u \quad \text{— (27)}$$

Let us compare evolution of vector $\bar{B}/\rho$ with evolution of fluid element

$$\frac{d}{dt}\,\delta\bar{r}(t) = \bar{u}(\bar{r}+\delta\bar{r}) - \bar{u}(\bar{r})$$

$$\approx \delta r\cdot\nabla\bar{u} \quad \text{— (28)}$$

Thus δr & $\bar{B}/\rho$ satisfy same eqn

∴ two fluid element starting on same field line will stay on the same line

$$\delta r = \text{const}\,\frac{\bar{B}}{\rho} \quad \text{--- (29)}$$

This in MHD the flow will be entraining magnetic field lines with it

- and the field line will react back on the fluid
- when fluid tries to bend field, field will want to spring back
- fluid tries to compress or rarify field will resist.

MHD fluid acts as elastic: it is threaded by magnetic field line which move with it & act as elastic bands

So far we have now

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \bar{u}) = 0 \quad -(30)$$

$$\rho\left(\frac{\partial \bar{u}}{\partial t} + \bar{u}\cdot\nabla\bar{u}\right) = -\bar{\nabla} p - \nabla\cdot\bar{\Pi} + \frac{(\nabla\times\bar{B})\times B}{4\pi}$$

$$= -\nabla\cdot\left[\left(p + \frac{B^2}{8\pi}\right)\mathbb{I} - \frac{\bar{B}\bar{B}}{4\pi} + \Pi\right] \quad -(31)$$

$$\frac{\partial \bar{B}}{\partial t} = \nabla\times(\bar{u}\times\bar{B}) + \eta\nabla^2\bar{B} \quad -(32)$$

To complete we need energy eqn.

Total energy density

$$\epsilon = \frac{\rho u^2}{2} + \frac{p}{\gamma-1} + \cancel{\frac{E^2}{8\pi}} + \frac{B^2}{8\pi} \qquad -(33)$$

kinetic internal Electric Magnetic

$E^2/B^2 << 1$

following the same logic as earlicar.

$$\underbrace{\frac{d}{dt}\int_V d^3\bar{r}\, \epsilon}_{\text{energy inside volum of fluid}} = -\underbrace{\int_{\partial V}\left(\frac{\rho u^2}{2} + \frac{p}{\gamma-1}\right)\bar{u}\cdot d\bar{s}}_{\text{kinetic + interna. carried by fluid volum}}$$

$$-\underbrace{\int_{\partial V}\{(p\,I + \Pi)\cdot\bar{u}\}\cdot ds}_{\text{work done on boundary by pressure + stress.}}$$

$$-\underbrace{\int_{\partial V} q\cdot ds}_{\text{heat flux}} - \underbrace{\int_{\partial V}\frac{c}{4\pi}\left(\bar{E}\times\bar{B}\right)\cdot d\bar{s}}_{\text{poynting flux.}} \qquad -(34)$$

$$\left(\frac{\partial}{\partial t} + \bar{u}\cdot\nabla\right)\frac{p}{\rho^\gamma} = 0 \qquad -(35)$$