Fluid Mechanics

Continuity Eq^n:

Consider $\bar{x} = (x, y, z)^T \in D$ is a point in D. At each time t, let the fluid has well defined mass density $\rho(\bar{x}, t)$ at a point $\bar{x}$. Each fluid particle traces out a well defined path in the fluid, and its motion along that path is governed by the velocity field $\bar{u}(\bar{x}, t)$ at position $\bar{x}$, at time t. Consider an arbitary subregion $\Omega \subseteq D$. The total mass of fluid contained inside this region Ω at time t is

$$\int_{\Omega} \rho(\bar{x}, t)\, dV$$

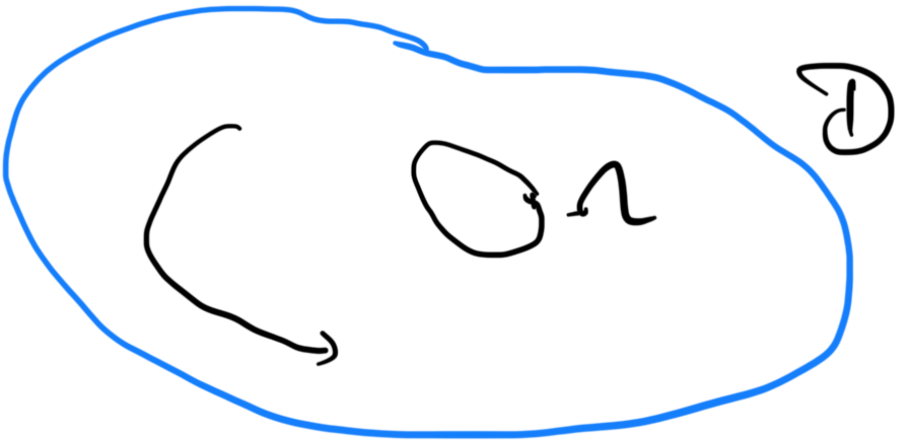
where dV is a volume element in $\mathbb{R}^d$. Let us consider rate of change of mass in Ω.

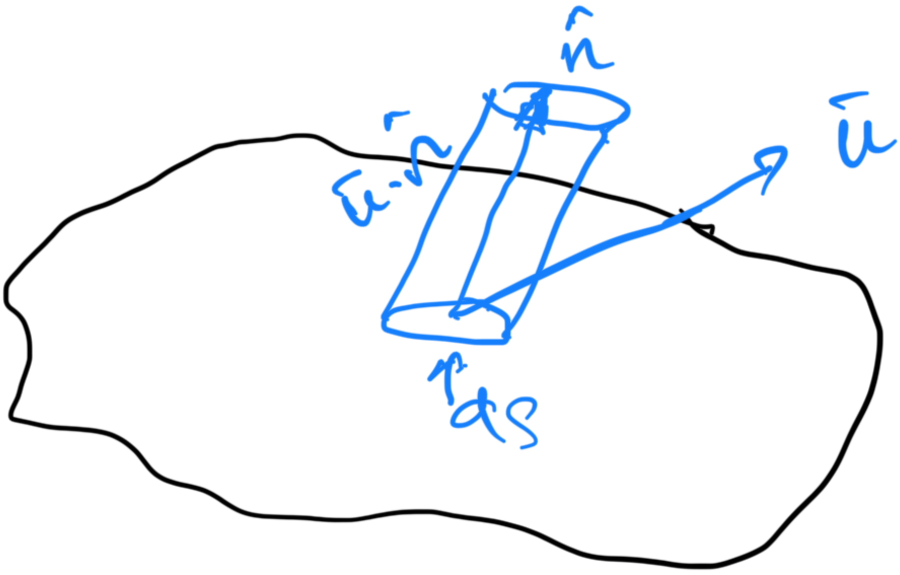
by principle of conservation of mass, the rate of increase of mass in Ω is given by the mass of fluid entering / leaving the boundary $\partial\Omega$ of Ω per unit time.

To compute the total mass of fluid entering / leaving the boundary $\partial\Omega$ per unit time, we consider small area dS on the boundary $\partial\Omega$, which has unit outward normal $\hat{n}$, then

mass density × fluid volume leaving per unit time $= \rho(\bar{x},t)\,\bar{u}(\bar{x},t)\cdot\hat{n}(\bar{x})\,dS$

where $\bar{x}$ is the center of the area dS on $\partial\Omega$





Thus the equivalently in the integral form this can be written as

$$\frac{d}{dt}\int_{\Omega} \rho(\bar{x},t)\, dV = -\int_{\partial\Omega} \rho \bar{u}\cdot \hat{n} ds$$

from divergence theorem

$$\int_{\Omega} \nabla\cdot(\rho \bar{u})\, dV = \int_{\partial\Omega} (\rho\bar{u})\cdot \hat{n}\, ds$$

and

$$\frac{d}{dt}\int_{\Omega} \rho\, dV = \int_{\Omega} \frac{\partial \rho}{\partial t} dV$$

These two relations gives

$$\int_{\Omega} \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\bar{u})\, dV = 0$$

since Ω is arbitrary we get

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\bar{u}) = 0$$

This is first law of conversation