Fluid description of Plasma

One of the building blocks in fluid description is mass & particle flux

we use continuity equation describing mass conservation.

- the second is the momentum equation or the force equation.

Fluid ⇒

Density ρ ⇒ mass density

$$\rho = n\, m$$

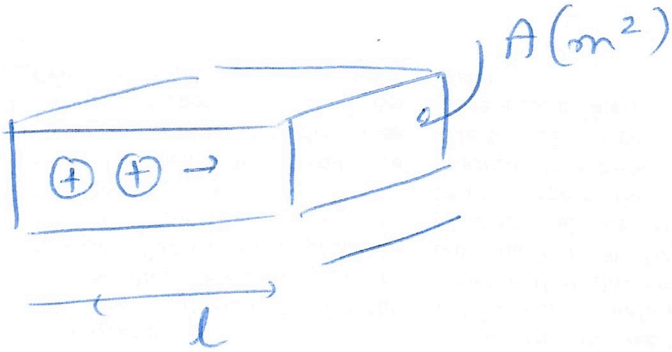
kg/m^3 ↑ number density ↑ mass of each particle

Pressure

Velocity — velocity of fluid element or bulk velocity

Temperature, T ⟶ Maxwellian

let have volume ~~ele~~ment of length l & area A (m^2)



number of particle in box = number density x Volume

$$N = n \times A l$$

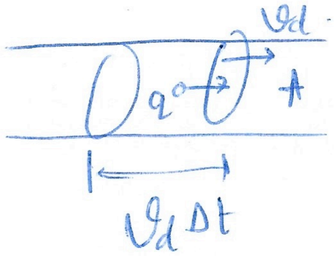
Number of particles leaving the box in time T

$$\frac{N}{T} = \frac{n A l}{T} \quad ; \quad \frac{l}{T} = v \Rightarrow \text{Speed of elect}$$

$$= n A v$$

Particle flux = $\dfrac{N}{TA} = n\vec{v}$ particle/m^2sec

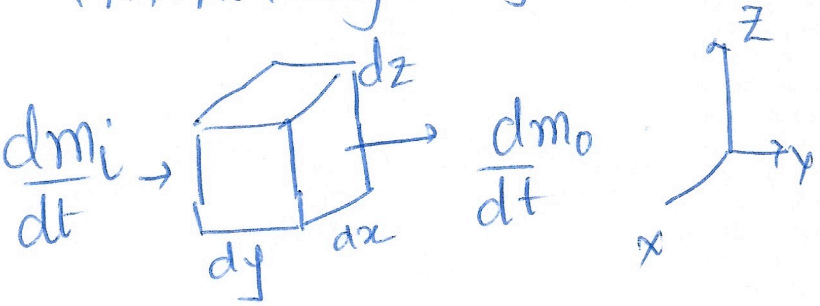
Mass flux = $mn\vec{v} = \rho\vec{v}$ kg/m^2-sec eg Current density



$$J = \frac{I}{A} = n q v_d$$

Charge flux

Consider fluid flow into & out of an infinitesimally small cube



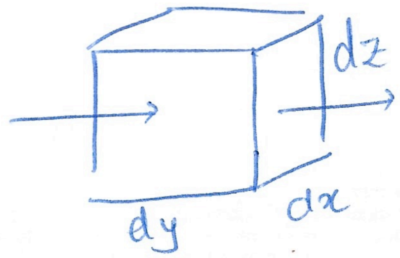
Rate of change of mass in the cube = mass flow rate out − mass flow rate in

$$\frac{dm}{dt} = \frac{dm_o}{dt} - dm_i$$

$$\frac{dm}{dt} = \text{mass flux} \times \text{Area}$$

$$\rho \vec{v} \times dx\,dz$$

before general expression let's look specific case.



$$\frac{dm_{yi}}{dt} = \rho_{y_i} v_{y_i} \, dx dz$$

$$\frac{dm_{y_o}}{dt} = \rho_{y_o} v_{y_o} \, dx dz$$

$$\frac{dm}{dt} = \frac{dm_o}{dt} - \frac{dm_i}{dt}$$

$$\frac{dm}{dt} = \left(\frac{dm_{xo}}{dt} + \frac{dm_{yo}}{dt} + \frac{dm_{zo}}{dt} \right) - \left(\frac{dm_{x_i}}{dt} + \frac{dm_{yi}}{dt} + \frac{dm_{zo}}{dt} \right)$$

$$\frac{\partial \rho}{dt} \, dx dy dz = \left(\rho_{x_o} v_{x_o} - \rho_{x_i} - \cancel{v}_{x_i} \right) dy \, dz$$
$$+ \left(\rho_{y_o} v_{y_o} - \rho_{y_i} v_{y_i} \right) dx \, d\cancel{z}$$
$$+ \left(\rho_{zo} v_{zo} - \rho_{zi} v_{zi} \right) dx \, dy.$$

$$- \Delta \rho_x v_x \rightarrow -ve$$

so that mass going out is less than coming in.

$$= d(\rho_x v_x)$$

unless compressible fluid.

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho_x v_x)}{\partial x} - \frac{\partial(\rho_y v_y)}{\partial y} - \frac{\partial(\rho_z v_z)}{\partial z}$$

$\therefore$ Continuity eqn.

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \vec{v}) = 0$$

Momentum eqn

How fluid velocity changes from (x, y, z) to $(x+\Delta x, y+\Delta y, z+\Delta z)$

$$v \equiv v(x, y, z, t)$$

$$\frac{d\vec{v}}{dt} = \frac{\partial \bar{v}}{\partial t} + (\bar{v}\cdot\nabla)\vec{v}$$

$$\rho\frac{d\bar{v}}{dt} = \rho\frac{\partial v}{\partial t} + \rho(v\cdot\nabla)\bar{v}$$

units force per unit Volume for each term.

$$= nq(\bar{E} + \bar{v}\times\bar{B}) - \bar{\nabla} p$$

$\uparrow$ Lorenz eqn.

$\uparrow$ pressure grad

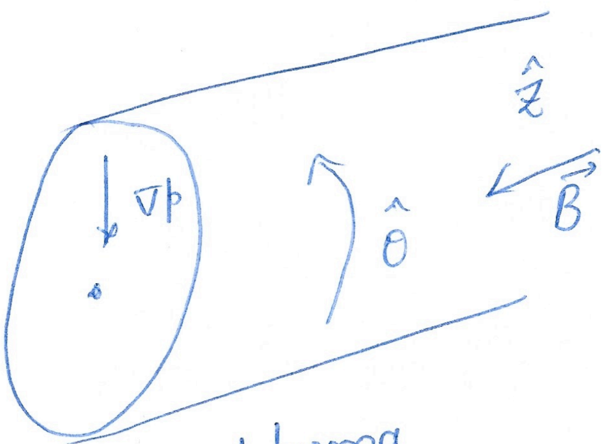
Two fluid plasma

let electrons & ions

$$\frac{\partial n_\alpha}{\partial t} + \nabla(n\vec{v}_\alpha) = 0$$

$$n_\alpha m_\alpha \left[\frac{\partial \vec{v}_\alpha}{\partial t} + \underbrace{(\vec{v}_\alpha \cdot \nabla)\vec{v}_\alpha}_{\text{convective term.}}\right] = n_\alpha q_\alpha\left[\vec{E} + \vec{v}_\alpha \times \vec{B}\right] - \nabla p_\alpha$$

α = ions, electrons



Assume $\vec{v} = v_\perp$

$\bar{v}_\perp$ is very slow compare to cyclotron motion

$$\frac{\partial v_\perp}{\partial t} = 0$$

$$(v_\perp \cdot \nabla)\bar{v}_\perp = 0$$

Since it is second order term in velocity

$$0 = qn[\bar{E} + (\bar{v}_\perp \times \bar{B})] - \nabla p$$

cross product with $\bar{B}$

$$0 = qn\left[(\bar{E}\times\bar{B}) + \underbrace{(v_\perp \times \bar{B})\times\bar{B}}_{(\bar{v}_\perp\cdot\bar{B})\bar{B} - v_\perp B^2}\right] - \bar{\nabla}p \times \bar{B}$$

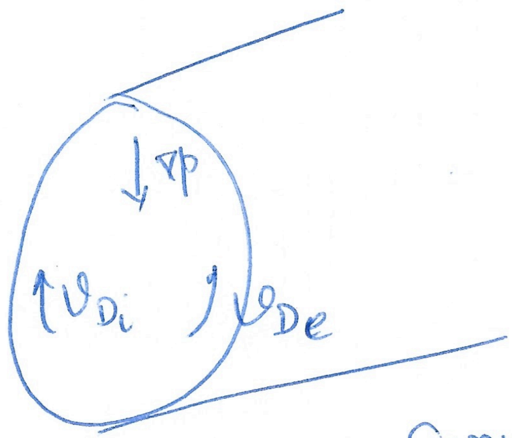
$$\underbrace{(\bar{v}_\perp \cdot \bar{B})}_{0}$$

$$v_\perp = \frac{\bar{E} \times \bar{B}}{\bar{B}^2} - \frac{\nabla p \times \bar{B}}{q n B^2}$$

$$v_\perp = v_E + v_D$$

$$v_D = -\frac{\nabla p \times \bar{B}}{\underline{q} n B^2}$$

charge $\Rightarrow$ diamagnetic drift Velocity



Convective term.

$$mn\left[\frac{\partial v_z}{\partial t} + (\bar{v} \cdot \nabla) v_z\right] = q v_z E_z - \frac{\partial p}{\partial z}$$

$\vec{v}$ is slow enough to ignore $(\bar{v} \cdot \bar{\nabla}) V_z$

But Fast enough not ignore $\frac{\partial v_z}{\partial t}$

$\partial p / \partial z$ using equ^n of state for ideal adiabatic gas.

$$p = C n^\gamma$$

$$p = C n^{\gamma}$$ ← ratio of Spcific heat $\left(\frac{C_p}{C_v}\right)$

↑ const. ↑ number density ↑ Const p, v.

$$\frac{\partial p}{dz} = C \gamma n^{\gamma - 1} \frac{\partial n}{\partial z}$$

divide left by p & right by Cn^{γ}

$$\frac{1}{p}\frac{\partial p}{\partial z} = \frac{\gamma}{n}\frac{\partial n}{\partial z}$$

Now we substitabe for ideal gas law

$$p = n k_B T$$

$$\frac{\partial p}{\partial z} = \frac{\gamma k_B T}{n}\frac{\partial n}{\partial z}$$

∴ momentum eqn

$$\frac{\partial v_z}{\partial z} = \frac{q}{m} E_z - \frac{\gamma k_B T}{mn}\frac{\partial n}{\partial z}$$

in absence of drift.

$$\frac{q}{m} E_z = \frac{\gamma k_B T}{mn}\frac{\partial n}{\partial z}$$

$$q E_z = e \frac{\partial \phi}{\partial z}$$ — potential.

assume $\gamma = 1$ for fully ionize plasma

$$e \frac{\partial \phi}{\partial z} = \frac{k_B T}{n} \frac{\partial n}{\partial z}$$

$$e\phi = k_B T \ln n + C$$

$\therefore$ on rearranging

$$n = n_0 \exp\left(e\phi / k_B T\right)$$

The Boltzmann Relation.

Kinetic theory.

distribultion fc^n generalize form

$f \quad \bar{r}, \bar{v}, t$

position velociy time

deals motion of individual particles.

not feasible to track — motion.

So statistical approach.

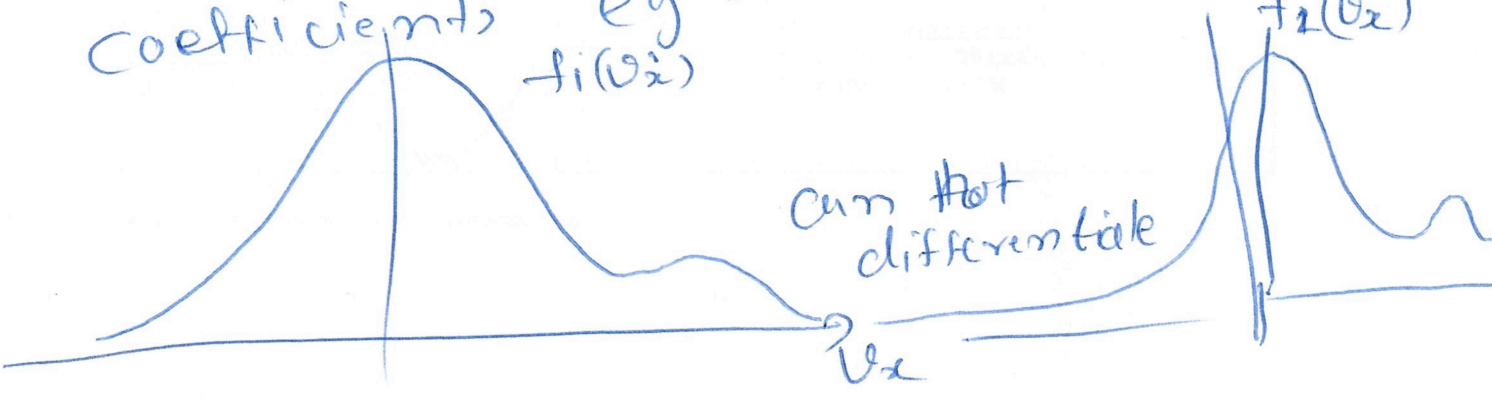
fluid model also hard x, y z, & t as independent variables.

Cant handle individual particles.

∴ Macroscopic parameters are used to describe plasma.

- density, pressure, temperature.

Kinetic theory is more relevant for discharge plasma to calculate reaction coefficients eg.



Kinetic theor

$$f(\bar{r}, \bar{v}, \hat{t})$$

7 independent parameters.

$$r = x\hat{i} + y\hat{j} + z\hat{k}$$

$$v = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

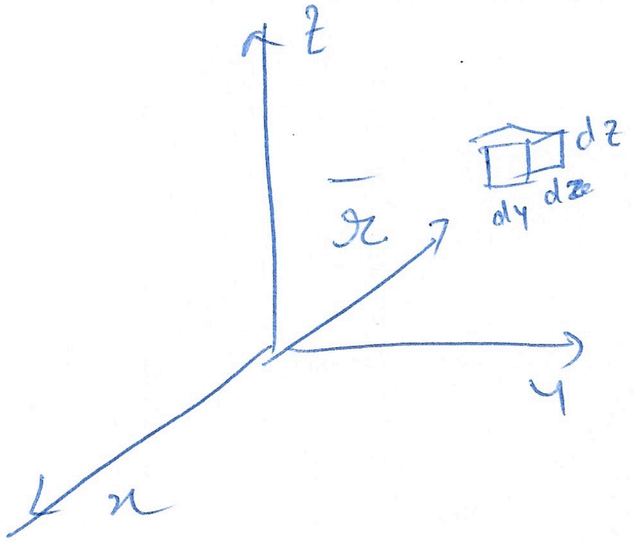
$(\bar{r}, \bar{v}) \Rightarrow$ define particle position in phase-space.

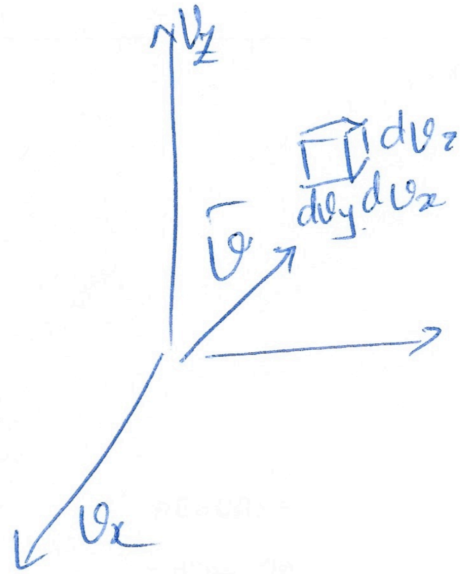
Volume element in phase-space

$$dV = d\bar{v}\, d\bar{r}.$$

$$d\bar{r} = d^3r \quad dx\, dy\, dz$$

$$d\bar{v} = d^3v = dv_x\, dv_y\, dv_z$$





Meaning of $f(\bar{r}, \bar{v}, t)$

$$f(\bar{r}, \bar{v}, t)\, d\bar{r}\, d\vec{v}$$

number of particles in volume element dV in phasespace

$$= dN(\bar{r}, \bar{v}, t)$$

f is the number density in phase space at time t

Particle number density in real space

$$n(\bar{r}, \bar{t}) = \int_{-\infty}^{+\infty} dv_x \int_{-\infty}^{+\infty} dv_y \int_{-\infty}^{\infty} dv_z\, f(\bar{r}, \bar{v}, t)$$

$$= \int_{-\infty}^{\infty} f(\bar{r}, \bar{v}, t)\, d\bar{v} \quad \text{shorter notation}$$

if $\hat{f}(\bar{r}, \bar{v}, t)$ is normalized version of $f(\bar{r}, \bar{v}, t)$ then

$$\int_{-\infty}^{+\infty} \hat{f}\, dv\, d\bar{r} = 1$$

If we take normalize distribution fu^n & multiply phase volume $dV = d\bar{r}\, dv \Rightarrow \hat{f}\, d\bar{v}\, d\bar{r}$ is probability of finding particle if $dV = d\bar{v}\, d\bar{r}$

Take away $d\bar{v}$ just normalize distr^n fu^n probability per unit volume & also called Probability Density.

Probability fu^n example.

let Room with 14 people ages:

1	person	14
1		15
3	people	16
2		22
2		24
5		25

$N(j) \Rightarrow$ number of people with age j

$N(14) = 1$

$N(15) = 1$

$N(16) = 3.$

$\vdots$

$N(25) = 5$

Total number

$$N = \sum N(j) = 14$$

Random selection. $\Rightarrow$ what is probability that person is aged 16?

$$P(16) = \frac{N(16)}{N} = \frac{3}{14}$$

$$P(j) = \frac{N(j)}{N}$$

$$\frac{\sum N(j)}{N} = 1$$

what is mean age

$$\frac{(14)+(15)+3(16)+2(22)\ldots}{14} = \frac{294}{14} = 21$$

weighted age.

avg age in general.

$$\bar{j} = \frac{\sum j N(j)}{N} = \sum j P(j)$$

$$\bar{j} = \int_0^\infty j\, P(j)\, dj \qquad \int_0^\infty P(j)\, dj = 1$$

Probability of finding particle at time t at posi $\vec{r}$ with velocity between v & $v + dv$

$$\hat{f}(\bar{r}, \bar{v}, t)\, d\bar{v}$$

normalize distribution fun

average speed.

$$\bar{v} = \int v\, \hat{f}(r, v, t)\, dt$$

leaving in liter

if system is collisional with freq^n ν

Then after time long compared to $1/\nu$

equipartition of energy will cause system

to move to Maxwellian with distribution

$$f_m = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(\frac{-v^2}{v_{Th}^2}\right)$$

$$v_{Th} = \left(\frac{2k_B T}{m}\right)^{1/2}$$ thermall velociy

if n = density

$$f(\bar{v}) = n\left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{v^2}{v_{Th}^2}\right)$$

$$\int d^3v \, f(\bar{v}) \Rightarrow \quad \underline{v^2 = v_x^2 + v_y^2 + v_z^2}$$

$$I_n = \int_0^\infty x^n e^{-ax^2} dx$$

$$I_0 = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$I_1 = \frac{1}{2a}$$

$$I_n = \frac{n-1}{2a} I_{n-2}$$

$$\bar{v} = \frac{1}{n}\int f d^3\bar{v} = \left(\frac{8k_B T}{\pi m}\right)^{1/2}.$$

$$f = n\left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{\left(-\frac{m\vartheta^2}{2k_B T}\right)}$$

$$\int f\, d^3\vartheta$$

$$= n\left(\frac{m}{2\pi kT}\right)^{3/2} \int_{-\infty}^{\infty} e^{-\frac{m\vartheta^2}{2KT}}$$

$$= n\left(\frac{m}{2\pi KT}\right)^{3/2} \left[\sqrt{\frac{\pi}{(m/2KT)}}\right]^3$$

$$= n\left(\frac{m}{2\pi KT}\right)^{3/2} \left(\frac{2\pi KT}{m}\right)^{3/2}$$

$$= n$$

$$\frac{1}{n}\int \vartheta f\, d^3\vartheta = n\left(\frac{m}{2\pi kT}\right)^{3/2} \int_{-\infty}^{-\infty} \vartheta e^{-\frac{m\vartheta^2}{2KT}}$$

$$f(\vartheta)d\vartheta = \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi \vartheta^2 e^{-m\vartheta^2/2kT} d\vartheta$$

$$b = \frac{m}{2kT}$$

$$|v| = \int_0^\infty v \, f(v) \, dv$$

$$= 4\pi \left(\frac{b}{\pi}\right)^{3/2} \int_0^\infty v^3 \, e^{-bv^2} \, dv$$

$$= 4\pi \left(\frac{b}{\pi}\right)^{3/2} \left(\frac{1}{2b^2}\right)$$

$$= \sqrt{\frac{4}{\pi b}}$$

$$= \sqrt{\frac{8kT}{\pi m}}$$

Consider small volume in 6 D phase space

$\int d^3\bar{x}\, d^3\bar{v}$; with surface area in real

space $\int d\bar{S}$ & in velocity space $\int d\bar{S}_{\bar{v}}$

Conservation of law rate of change of

particles in volume is equal to net

flux of particles into volume.

In $\bar{x}$-space flux.

$$\int f\, \dot{\bar{x}} \cdot d\bar{S} = \int f\, \bar{v} \cdot d\bar{S}$$

in $\bar{v}$-space the flux is

$$\int f\, \dot{v}\, d\bar{S}_{v} = \int f\, \bar{a} \cdot d\bar{S}_{v}$$

$$\frac{\partial}{\partial t} \int f\, d^3\bar{x}\, d^3v = -\int f\, v\, ds\, d^3v - \int f\, \bar{a} \cdot d\bar{S}_{v}\, d^3x$$

Gauss law $\oint\!\!\oint_S \bar{F} \cdot \bar{n}\, ds = \iiint_V \nabla \cdot \bar{F}\, dV$

$$\frac{\partial}{\partial t} \int f\, d^3\bar{x}\, d^3\bar{v} = -\int \frac{\partial}{\partial \underline{x}} \bar{v} f\, d^3\bar{x}\, d^3\bar{v}$$

$$- \int \frac{\partial}{\partial \bar{v}} (\bar{a} \cdot f)\, d^3x\, d^3v$$

div.

Since volume elements are arbitrary small

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \bar{x}}(\bar{v} f) + \frac{\partial}{\partial v}(\bar{a} f) = 0$$

Since $\bar{x}$, $\bar{v}$ are independn

$$\frac{\partial f}{\partial t} + \bar{v} \cdot \frac{\partial f}{\partial x} + \underbrace{\bar{a}}_{\Downarrow\ \frac{\vec{F}}{m}} \cdot \frac{\partial f}{\partial \bar{v}} = 0$$

This is called collisionless Boltzmann eqn

$$F = m\bar{a} = q(\bar{E} + \bar{v} \times \bar{B})$$

$$\frac{\partial f}{\partial t} + \bar{v}\frac{\partial f}{\partial \bar{x}} + \frac{q}{m}(\bar{E} + \bar{v} \times \bar{B})\frac{\partial f}{\partial \bar{v}} = 0$$

Vlasov eqn.

maxwell distn fun

$$\left(\frac{\partial f}{\partial t}\right)_c = -\left(\frac{f - f_m}{\tau}\right)$$

↳ mean collision time

Bhatnager Gross Krook

if masses are very diff eq electron-neutal.

$$\tau \rightarrow (m_n / 2m_e)\tau$$

Macroscopic eq^n of motion for electrons with collision

$$m_e \frac{D u_e}{Dt} = -e(\bar{E} + \bar{u}_e \times \bar{B}) + (F_{co})_e.$$

$$(F_c)_e = -\nu_c m_e u_e$$

ν_c frequency collision

$$m_e \frac{D \bar{u}_e}{Dt} = -e(\bar{E} + \bar{u}_e \times \bar{B}) - \nu_c m_e \bar{u}_e$$

in the absence of $\bar{E}$, $\bar{B}$

$$\frac{D \bar{u}_e}{Dt} = -\nu \bar{u}_e$$

$$\bar{u}_e(t) = \bar{u}_e(0) \exp(-\nu_c t)$$

Theis electron-neutral collision decrease average electron velocity exponentially with rate given by ν_c.

Similarly for $\bar{u}_i \Rightarrow$ ions.

we can neglect ion motion & $\bar{u}_i = 0$

$m_i >> m_e$

This is Lorentz gas.

Isotropic Plasma. DC case.

in the absence of $\vec{B}$

$$-e\bar{E} - m_e \nu_c \bar{u}_e = 0$$

$$\vec{J} = -e n_e \bar{u}_e$$ → electric current density

$$\bar{J} = \frac{n_e e^2}{m_e \nu_c} \vec{E}$$

Ohm's Law

$$\sigma = \frac{n_e e^2}{m_e \nu_c}$$

Electron mobility

$$M_e = \frac{u_e}{E} = \frac{-e}{m_e \nu_c} = -\frac{\sigma_0}{n_e e}$$